Applied Mathematics Letters 24 (2011) 1477–1480

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Alon–Babai–Suzuki's inequalities, Frankl–Wilson type theorem and multilinear polynomials

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ARTICLE INFO

Article history:
Received 2 March 2010
Received in revised form 16 December 2010
Accepted 17 December 2010

Keywords:
Alon–Babai–Suzuki's inequalities
Frankl–Wilson theorem
Multilinear polynomials

ABSTRACT

Let $K = \{k_1, k_2, \ldots, k_r\}$ and $L = \{l_1, l_2, \ldots, l_s\}$ be subsets of $\{0, 1, \ldots, p-1\}$ such that $K \cap L = \emptyset$, where p is a prime. Let $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ be a family of subsets of $[n] = \{1, 2, \ldots, n\}$ with $|F_i| \pmod p \in K$ for all $F_i \in \mathcal{F}$ and $|F_i \cap F_j| \pmod p \in L$ for any $i \neq j$. Every subset F_i of $[n]$ can be represented by a binary code $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ such that $a_j = 1$ if $j \in F_i$ and $a_j = 0$ if $j \notin F_i$. Alon–Babai–Suzuki proved in non-modular version that if $k_i \geq s - r + 1$ for all i, then $|\mathcal{F}| \leq \sum_{i=s-r+1}^{s} \binom{n}{i}$. We generalize it in modular version. Alon–Babai–Suzuki also proved that the above bound still holds under $r(s-r+1) \leq p-1$ and $n \geq s + \max_i k_i$ in modular version. Alon–Babai–Suzuki made a conjecture that if they drop one condition $r(s-r+1) \leq p-1$ among $r(s-r+1) \leq p-1$ and $n \geq s + \max_i k_i$, then the above bound holds. But we prove the same bound under dropping the opposite condition $n \geq s + \max_i k_i$. So we prove the same bound under only condition $r(s-r+1) \leq p-1$. This is a generalization of Frankl–Wilson theorem (Frankl and Wilson, 1981 [2]).

1. Introduction

In this paper, $\mathcal{F}$ stands for a family of subsets of $[n] = \{1, 2, \ldots, n\}$, $K = \{k_1, \ldots, k_r\}$ and $L = \{l_1, \ldots, l_s\}$ where $|F_i| \in K$ for all $F_i \in \mathcal{F}$, $|F_i \cap F_j| \in L$ for all $F_i, F_j \in \mathcal{F}$, $i \neq j$. The variable x will stand as a short-hand for the n-dimensional vector variable $(x_1, x_2, \ldots, x_n)$. Also, since these variables will take the values only 0 and 1, all the polynomials we will work with will be reduced modulo relation $x_i^2 = x_i$. We define the characteristic vector $v_i = (v_{i1}, v_{i2}, \ldots, v_{in})$ of F_i such that $v_{ij} = 1$ if $j \in F_i$ and $v_{ij} = 0$ if $j \notin F_i$. We will present some results in this paper that give upper bounds on the size of $\mathcal{F}$ under various conditions. Below is a list of related results by others.

Theorem 1 (*Ray-Chaudhuri and Wilson [1]*). *If $K = \{k\}$, and L is any set of nonnegative integers with $k > \max l_j$ for every j, then $|\mathcal{F}| \leq \binom{n}{s}$.*

Theorem 2 (*Frankl–Wilson [2]*). *Let L be a set of s integers and p a prime number. Assume $\mathcal{F}$ is a family of subsets of $[n]$ such that $|F_i| = k \pmod p \notin L$ for all i, and $|F_i \cap F_j| \pmod p \in L$ for $i \neq j$. Then $|\mathcal{F}| \leq \binom{n}{s}$.*

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0893-9659/$ – see front matter
doi:10.1016/j.aml.2010.12.014

Theorem 3 (*Alon et al. [3]*). *If K and L are two sets of nonnegative integers with $k_i > s - r$ for every i, then* $|\mathcal{F}| \le \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}$.

Theorem 4 (*Snevily [4]*). *If K and L are any sets such that $\min k_i > \max l_j$ for all i, j, then* $|\mathcal{F}| \le \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{0}$.

Theorem 5 (*Snevily [5]*). *Let K and L be sets of nonnegative integers such that $\min k_i > \max l_j$ for all i, j. Then* $|\mathcal{F}| \le \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-2r+1}$.

Snevily made the following conjecture.

Conjecture 6 (*Snevily [6]*). *For any K and L with $\min k_i > \max l_j$ for all i, j,* $|\mathcal{F}| \le \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-r}$.

Hwang and Sheikh [7] proved the bound of Conjecture 6 when K is a consecutive set. The second theorem we prove is a special case of Conjecture 6 with the extra condition that $\bigcap_{i=1}^m F_i \neq \emptyset$. These two theorems are stated hereunder.

Theorem 7 (*Hwang and Sheikh [7]*). *Let $K = \{k_1, k_2, \ldots, k_r\}$ where $k_i = k_1+i-1$, $k_1 > s-r$, and $L = \{l_1, l_2, \ldots, l_s\}$. Let $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ be such that $|F_i| \in K$ for each i, $|F_i| \notin L$, and $|F_i \cap F_j| \in L$ for any $i \neq j$. Then* $|\mathcal{F}| \le \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-r}$.

Theorem 8 (*Hwang and Sheikh [7]*). *Let $K = \{k_1, k_2, \ldots, k_r\}$, $L = \{l_1, l_2, \ldots, l_s\}$, and $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ be such that $|F_i| \in K$ for each i, $|F_i \cap F_j| \in L$ for any $i \neq j$, and $k_i > s - r$. If $\bigcap_{i=1}^m F_i \neq \emptyset$, then* $|\mathcal{F}| \le \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-r}$.

In 1991, Alon et al. proved the following theorems.

Theorem 9 (*Alon et al. [3]*). *Let K and L be sets of nonnegative integers and $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ be a family of subsets of $[n]$ such that $|F_i| \in K$ for all $F_i \in \mathcal{F}$ and $|F_i \cap F_j| \in L$ for $i \neq j$. If $k_i \ge s - r + 1$ for all i, then* $|\mathcal{F}| \le \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}$.

Theorem 10 (*Alon et al. [3]*). *Let K and L be subsets of $\{0, 1, \ldots, p - 1\}$ such that $K \cap L = \emptyset$, where p is a prime and $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ be a family of subsets of $[n]$ such that $|F_i| \pmod p \in K$ for all $F_i \in \mathcal{F}$ and $|F_i \cap F_j| \pmod p \in L$ for $i \neq j$. If $r(s - r + 1) \le p - 1$, and $n \ge s + \max k_i$ for every i, then* $|\mathcal{F}| \le \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}$.

The following theorem Hwang et al. proved is Alon–Babai–Suzuki's conjecture in non-modular version.

Theorem 11 (*Hwang et al. [8]*). *Let $K = \{k_1, k_2, \ldots, k_r\}$, $L = \{l_1, l_2, \ldots, l_s\}$ be two sets of nonnegative integers and $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ be such that $|F_i| \in K$ for each i, $|F_i \cap F_j| \in L$ for any $i \neq j$, and $n \ge s + \max k_i$ for every i, then* $|\mathcal{F}| \le \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}$.

Conjecture 12 (*Alon et al. [3]*). *Let K and L be subsets of $\{0, 1, \ldots, p - 1\}$ such that $K \cap L = \emptyset$, where p is a prime and $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ be a family of subsets of $[n]$ such that $|F_i| \pmod p \in K$ for all $F_i \in \mathcal{F}$ and $|F_i \cap F_j| \pmod p \in L$ for $i \neq j$. If $n \ge s + \max k_i$ for every i, then* $|\mathcal{F}| \le \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}$.

In [3], Alon et al. proved their conjectured bound under the extra conditions that $r(s-r+1) \le p-1$ and $n \ge s+\max_i k_i$ in modular version. Qian and Ray-Chaudhuri [9] proved that if $n \ge 2s-r$ instead of $r(s-r+1) \le p-1$ and $n \ge s+\max_i k_i$ in modular version, then the above bound holds. We prove the same bound with bound of Alon–Babai–Suzuki's conjecture when $r(s - r + 1) \le p - 1$ instead of $r(s - r + 1) \le p - 1$ and $n \ge s + \max_i k_i$ in modular version. Our Theorem 15 is a generalization of Frankl–Wilson theorem [2]. In the case of $r = 1$, we get Frankl–Wilson theorem [2]. We also get Theorem 13 which is a generalization of Alon–Babai–Suzuki's Theorem 9 to modular version.

Theorem 13. *Let $K = \{k_1, k_2, \ldots, k_r\}$ and $L = \{l_1, l_2, \ldots, l_s\}$ be subsets of $\{0, 1, \ldots, p-1\}$ such that $K \cap L = \emptyset$, where p is a prime and $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ be a family of subsets of $[n]$ such that $|F_i| \pmod p \in K$ for all $F_i \in \mathcal{F}$ and $|F_i \cap F_j| \pmod p \in L$ for $i \neq j$. If $k_i \ge s - r + 1$ for all i, then* $|\mathcal{F}| \le \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}$.

Proposition 14. *Let $K = \{k_1, k_2, \ldots, k_r\}$ and $L = \{l_1, l_2, \ldots, l_s\}$ be subsets of $\{0, 1, \ldots, p-1\}$ such that $K \cap L = \emptyset$, where p is a prime and $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ be a family of subsets of $[n]$ such that $|F_i| \pmod p \in K$ for all $F_i \in \mathcal{F}$ and $|F_i \cap F_j| \pmod p \in L$ for $i \neq j$. If $2s - 2r < n$, $n - k_i \le s$ and $k_i \le s - r$ for all i, then* $|\mathcal{F}| \le \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}$.

Theorem 15. *Let $K = \{k_1, k_2, \ldots, k_r\}$ and $L = \{l_1, l_2, \ldots, l_s\}$ be subsets of $\{0, 1, \ldots, p-1\}$ such that $K \cap L = \emptyset$, where p is a prime and $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ be a family of subsets of $[n]$ such that $|F_i| \pmod p \in K$ for all $F_i \in \mathcal{F}$ and $|F_i \cap F_j| \pmod p \in L$ for $i \neq j$. If $r(s - r + 1) \le p - 1$, then* $|\mathcal{F}| \le \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}$.

2. Proof of theorems

We recall the definition of gaps.

Definition 16 ([3]). A set $H = \{h_1, h_2, \ldots, h_m\} \subseteq [n]$ has a gap of size $\geq k$ (where the h_i are arranged in increasing order), if either $h_1 \geq k-1$, or $n - h_m \geq k-1$, or $h_{i+1} - h_i \geq k$ for some i $(1 \leq i \leq m-1)$.

Then we present the following lemma which is Lemma 3.6 in [3].

Lemma 17 ([3]). *Let $K \subseteq \{0, 1, \ldots, p-1\}$ be a set of integers and assume the set $H = (K + p\mathbb{Z}) \cap [n]$ has a gap $\geq s+1$, where $s > 0$. Let g denote the polynomial in n variables*

$$g(x) = \prod_{k \in K} \left(\sum_{t=1}^{n} x_t - k \right) \prod_{j \in I} x_j. \tag{1}$$

Then the set of polynomials $\{g(x) : |I| \leq s-1\}$ is linearly independent over $\mathbb{F}_p$.

Proof of Theorem 13. For each $F_i \in \mathcal{F}$, consider the polynomial

$$f_i(x) = \prod_{j=1}^{s} (v_i \cdot x - l_j),$$

where v_i is the characteristic vector of F_i. We order $\{F_i\}$ by size of F_i, that is, $|F_j| \leq |F_k|$ if $j < k$. Clearly, $f_i(v_i) \neq 0 (\text{mod } p)$ for $1 \leq i \leq m$ and $f_i(v_j) = 0 (\text{mod } p)$ for $1 \leq j \neq i \leq m$.

First, we prove that $\{f_i(x)\}$ is linearly independent over finite field $\mathbb{F}_p$. Assume that

$$\sum_i \alpha_i f_i(x) = 0.$$

Suppose that this is false. Let i_0 be the smallest index such that $\alpha_{i_0} \neq 0$. We substitute v_{i_0} into the above equation. Then we get $\alpha_{i_0} f_{i_0}(v_{i_0}) = 0 (\text{mod } p)$. Since $f_{i_0}(v_{i_0}) \neq 0 (\text{mod } p)$, we have $\alpha_{i_0} = 0$. So we get a contradiction. Thus, $\{f_i(x)\}$ is linearly independent over $\mathbb{F}_p$.

Let $\mathcal{E} = \{E_1, E_2, \ldots, E_e\}$ be the family of subsets of $[n]$ with size at most $s-r$, which is ordered by size, that is, $|E_i| \leq |E_j|$ if $i < j$, where $e = \sum_{i=0}^{s-r} \binom{n}{i}$. Let u_i denote the characteristic vector of E_i. We define the multilinear polynomial g_i in n variables for each E_i:

$$g_i(x) = \prod_{l=1}^{r} \left(\sum_{t=1}^{n} x_t - k_l \right) \prod_{j \in E_i} x_j.$$

We prove that $\{g_i(x)\}$ is linearly independent over $\mathbb{F}_p$. Since $s-r+1 \leq k_i$ for all i, we have that the set H has a gap $\geq s-r+2$. Hence, $\{g_i(x)\}$ is linearly independent over $\mathbb{F}_p$ by Lemma 17.

Next, we prove that $\{f_i(x), g_i(x)\}$ is linearly independent over $\mathbb{F}_p$. Now, assume that

$$\sum_i \alpha_i f_i(x) + \sum_i \beta_i g_i(x) = 0.$$

We substitute the characteristic vector v_i of the F_i into the above equation. Since $f_i(v_i) \neq 0$, we get $\alpha_i = 0$.

Any polynomial in the set $\{f_i(x), g_i(x)\}$ can be represented by a linear combination of multilinear monomials of degree $\leq s$. The space of such multilinear polynomials has dimension $\sum_{i=0}^{s} \binom{n}{i}$ by Grobner basis. We found $|\mathcal{F}| + \sum_{i=0}^{s-r} \binom{n}{i}$ linearly independent polynomials with degree at most s. So $|\mathcal{F}| + \sum_{i=0}^{s-r} \binom{n}{i} \leq \sum_{i=0}^{s} \binom{n}{i}$. Thus $|\mathcal{F}| \leq \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}$. $\square$

Proof of Proposition 14. For each $F_i \in \mathcal{F}$, consider the polynomial

$$f_i(x) = \prod_{j=1}^{s} (v_i \cdot x - l_j),$$

where v_i is the characteristic vector of F_i. By the same way of proof of Theorem 13, $\{f_i(x)\}$ is linearly independent over $\mathbb{F}_p$.

Let $\mathcal{E} = \{E_1, E_2, \ldots, E_e\}$ be the family of subsets of $[n]$ with size at most $s-r$, which is ordered by size, that is, $|E_i| \leq |E_j|$ if $i < j$, where $e = \sum_{i=0}^{s-r} \binom{n}{i}$. Let u_i denote the characteristic vector of E_i. We define the multilinear polynomial g_i in n variables for each E_i:

$$g_i(x) = \prod_{l=1}^{r} \left(\sum_{t=1}^{n} x_t - k_l \right) \prod_{j \in E_i} x_j.$$

We prove that $\{g_i(x)\}$ is linearly independent over $\mathbb{F}_p$. We claim that the set H has a gap $\geq s-r+2$. Since $n-k_i < s$ for all i, we have that $n < p+k_i$ from $s \leq p-1$ for all i. Thus, the set H consist of only $\{k_1, \ldots, k_r\}$. Since $2s-2r < n$ and $k_i \leq s-r$ for all i, we have that $s-r = (2s-2r)-(s-r) < n-(s-r) \leq n-k_i$ for all i. Thus, the set H has a gap $\geq s-r+2$. Hence, by Lemma 17, $\{g_i(x)\}$ is linearly independent over $\mathbb{F}_p$.

Next, we prove that $\{f_i(x), g_i(x)\}$ is linearly independent over $\mathbb{F}_p$. Now, assume that

$$\sum_i \alpha_i f_i(x) + \sum_i \beta_i g_i(x) = 0.$$

We substitute the characteristic vector v_i of the F_i into the above equation. Since $f(v_i) = 0$, we get $\alpha_i = 0$. Thus, $\{f_i(x), g_i(x)\}$ is linearly independent over $\mathbb{F}_p$. So by the same way of proof of Theorem 13, we get bound $|\mathcal{F}| \leq \binom{n}{s}+\binom{n}{s-1}+\cdots+\binom{n}{s-r+1}$. $\square$

Proof of Theorem 15. First, we prove theorem for $r \geq 2$. Our original proof for this theorem used the polynomial method in just the same way as we proved Proposition 14. Alon–Babi–Suzuki proved the bound under $r(s-r+1) \leq p-1$ and $n-k_i \geq s$. So the remaining proof is only for $r(s-r+1) \leq p-1$ and $n-k_i < s$ for $r \geq 2$. By Theorem 13, we only need to consider in the case of $k_i \leq s-r$. That is, we only need to consider in the case of $k_i \leq s-r$, $r(s-r+1) \leq p-1$ and $n-k_i < s$ with $r \geq 2$. Since $r(s-r+1) \leq p-1$ for $r \geq 2$, then $2(s-r) < 2(s-r+1) \leq r(s-r+1) \leq p-1 < n$. Hence, we have that $2s-2r < n$. Thus, we get conditions $k_i \leq s-r$, $2s-2r < n$ and $n-k_i < s$ which imply the conditions of above Proposition 14. By Proposition 14, we proved the bound. In the case of $r = 1$, Frankl and Wilson proved this in Theorem 2. $\square$

3. Remark

Alon et al. proved Theorem 9 when the conditions $r(s-r+1) \leq p-1$ and $n+\max k_i \leq n$ for all i. In the same paper, they conjecture that if we drop the condition $r(s-r+1) \leq p-1$, then the above bound still holds. In [9], Qian and Ray-Chaudhuri proved that with the condition $n \geq 2s-r$ instead of $r(s-r+1) \leq p-1$ and $s+\max k_i \leq n$ for all i, the same bound holds. Now we make the following theorems: if $k_i \geq s-r+1$ for all i, then the above bound also holds which is modular version of [3]. Instead of their conjecture, we prove their bound if $r(s-r+1) \leq p-1$.

Acknowledgement

This work was supported by the Dong-A university research fund.

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